SPERNER'S LEMMA AND BROUWER'S FIXED POINT THEOREM

ALEX WRIGHT

1. Intoduction

A fixed point of a function f from a set X into itself is a point x_0 satisfying $f(x_0) = x_0$.

Theorems which establish the existence of fixed points are very useful in analysis. One of the most famous is due to Brouwer. Let
$D_n = \{x \in \mathbb{R} : |x| \leq 1\}$.

Theorem (Brouwer's Fixed Point Theorem.)**.** *Every continuous function $f : D_n \to D_n$ has a fixed point.*

The purpose of this note is to provide a proof of Brouwer's Fixed Point Theorem for $n = 2$ using a combinatorial result know as Sperner's Lemma, and to explore both Sperner's Lemma and Brouwer's Fixed Point Theorem. We begin by examining the case where $n = 1$.

Proof for $n = 1$. We have $f : [-1, 1] \to [-1, 1]$. Define $g(x) = x - f(x)$. Then $g(1) \geq 0$ and $g(-1) \leq 0$ so by the Intermediate Value Theorem there must be a point x_0 so that $g(x_0) = 0$. We then have $f(x_0) = x_0$. □

Definition. *A fixed point domain is a set $G \subset \mathbb{R}^n$ such that every continuous function from G to itself has a fixed point.*

So Brouwer's Fixed Point Theorem asserts that each D_n, $n \geq 1$, is a fixed point domain.

On the other hand, D_n less a point, $[0, 1] \cup [2, 3]$ and $\mathbb{R}$ are not fixed point domains. (It is easy to construct appropriate functions with no fixed point for these sets.) The fact that D_n is compact is important in Brouwer's Fixed Point Theorem.

However, if G is a fixed point domain, and $f : G \to H$ is a continuous bijection with a continuous inverse, then H is a fixed point domain.

Proof. If $g : H \to H$ is continuous, so is $f^{-1} \circ g \circ f$. As this is a continuous function from G to G, it has a fixed point x_0. Now $(f^{-1} \circ$

Date: July 2005.

$g \circ f)(x_0) = x_0$, so $g(f(x_0)) = f(x_0)$, which means $f(x_0)$ is a fixed point of g. □

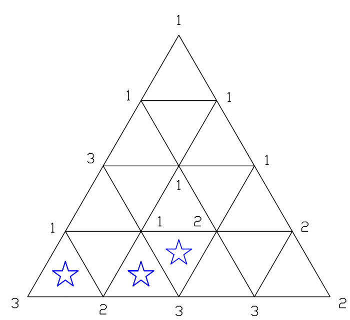
This means that being a fixed point domain is a topological property. Specifically, Brouwer's Fixed Point Theorem for $n = 2$ is now equivalent to showing that every continuous function from a triangle to itself has a fixed point because there is a continuous bijection with a continuous inverse from the circle to a triangle.

2. Sperner's Lemma

A triangulation of a triangle is a subdivision of the triangle into small triangles. The small triangles are called baby triangles, and the corners of the triangles are called vertices. The additional condition that each edge between two vertices is part of at most two triangles is necessary.

A Sperner Labeling is a labeling of a triangulation of a triangle with the numbers 1,2 and 3 such that

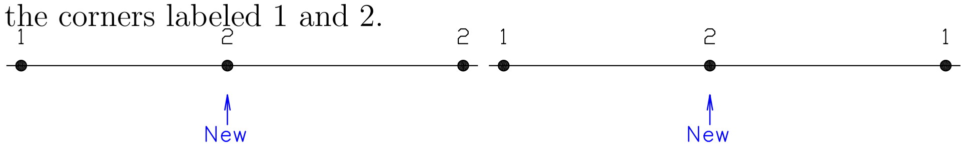
(1) The three corners are labeled 1, 2 and 3.
(2) Every vertex on the line connected vertex i and vertex j is labeled i or j.



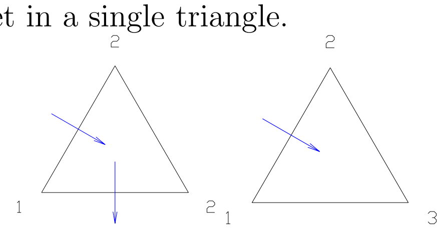
Lemma (Sperner's Lemma). *Every Sperner Labeling contains a baby triangle labeled 1-2-3.*

Sperner's Lemma generalizes to n dimensions, were it says that every coloring triangulation of a hyper tetrahedron into $n + 1$ colours must have an baby hyper tetrahedron that is labeled with all $n + 1$ labels. The proof that follows in fact generalizes to n dimensions too. For details, see [4].

Proof. First, we need the simple fact that the number of 1-2 edges on the outside of the triangle is odd. This can be seen by induction on the number of points on the edge between the 1 corner and the 2 corner. When there are no points in between the 1 and 2 corners, we have only this single 1-2 edge itself. If we add a 1 point in between a 1 and a 2 (or vice-versa), we get no net change. If we add a 1 point in between two 2 points (or vice versa), we get a net change of two 1-2 edges. So we start with one 1-2 edge, and can only make net changes of 2, which proves that there are an odd number of 1-2 edges on the line connecting the corners labeled 1 and 2.

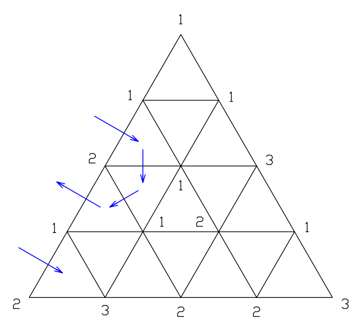


We now view the triangle as a house. Each baby triangle is a room, and each 1-2 edge is a door. We consider a path through the triangle starting from outside the triangle. Note that such a path can only end by leaving the triangle again, or by entering a 1-2-3 triangle. Note also that once the entrance is chosen, the path is uniquely determined because no triangle can have three 1-2 edges. This also means that no two paths can meet in a single triangle.



Consider all such paths. A path which ends by exiting the triangle determines a pair of 1-2 edges. Such paths account for an even number of 1-2 edges on the boundary. Since the number of 1-2 edges is odd, there must be at least one 1-2 edge whose path terminates in a 1-2-3 triangle.

□



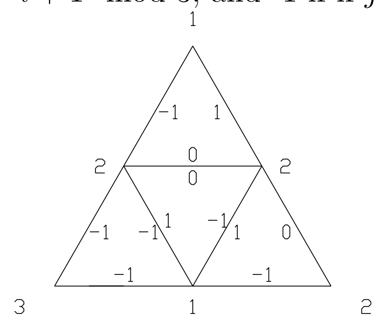
Lemma (Stronger form of Sperner). *Without loss of generality, suppose that the outside triangle in a Sperner labeling is labeled 1-2-3 in clockwise order. Let A be the number of baby 1-2-3 triangles oriented in the clockwise direction, and let B be the number of such triangles oriented in the counter clockwise direction. Then $A - B = 1$.*

Note that if we prove this, we have that $A+B$, which is total number of baby 1-2-3 triangles, is odd, and thus greater than 0.

The last proof of Sperner's Lemma can be modified to show this stronger result. The interested reader should try to make this modification.

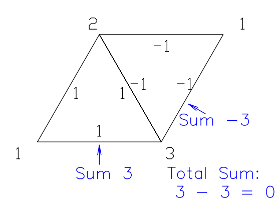
This result is included in addition to the last proof of Sperner's Lemma because the proofs are beautiful, insightful, and significantly different. This next proof is from [2]. For yet another beautiful proof that generalizes easily to a higher dimensions, see [3, p.38].

Proof. We will label each edge in the triangulation. If an edge has a triangle on either side, it will get two labels, one for each baby triangle it is part of. If we have a baby triangle, and two of its vertices (in clockwise direction) are labeled i and j, the label of the edge is 0 if $i = j \mod 3$, 1 if $j = i + 1 \mod 3$, and -1 if if $j = i - 1 \mod 3$.



We make three important observations.

(1) The inside edge labeling of a baby triangle sum to 3 if the triangle is 1-2-3 clockwise, -3 if it is 1-2-3 counterclockwise, and 0 otherwise.
(2) If an edge participates in two triangles, then its label in one will be the negative of its label in the other.
(3) If we consider the sum of the inside edge labels of a polygon composed of baby triangles, we get the sum of the sums of the inside edge labels of the baby triangles.



The sum of the edges of the big triangle is 3, because each side contributes 1 in total. Since this is also equal to the sum of the sums of the inside edge labels of all the baby triangles, we get

$$3A - 3B = 3$$

□

3. Barycentric Coordinates

Barycentric coordinates express a point in a triangle as a weighted average of the three vertices. Barycentric coordinates are thus always ordered triples (each weight is one number) that add up to 1.

For example, if the triangle is $(0,0),(0,1),(1,0)$, the Barycentric coordinates of the three vertices are $(1,0,0),(0,1,0)$ and $(0,0,1)$ respectively. In this example, the Barycentric coordinates of the point $(\frac{1}{4},\frac{1}{4})$ would be $(\frac{1}{2},\frac{1}{4},\frac{1}{4})$.

Barycentric coordinates always exist, because a triangle can be thought of as the convex hull its vertices.

4. Brouwer's Fixed Point Theorem

Theorem (Brouwer's Fixed Point Theorem for $n = 2$)**.** *Every continuous function from a triangle to itself has a fixed point.*

Although we will not prove Brouwer for $n > 2$, it is worth noting that both Sperner's Lemma and the techniques of this proof generalize to higher dimensions. The proof that follows is from [1].

Proof. Let f be a continuous function from a triangle T into itself. We will write $(a,b,c) \mapsto (a',b',c')$ is $f(a,b,c) = (a',b',c')$ in Barycentric coordinates.

We will label each point in the triangle as follows. Suppose $(a,b,c) \mapsto (a',b',c')$.

- If $a' < a$, we label (a,b,c) 1.
- If $a' \geq a$, but $b' < b$ we label (a,b,c) 2.
- If $a' \geq a$ and $b' \geq b$, but $c' < c$ and we label (a,b,c) 3.

If we cannot label the point (a,b,c), then $a' \geq a$, $b' \geq b$ and $c' \geq c$ so $a = a'$, $b = b'$, $c = c'$ and the point is a fixed point. For the remainder of the proof, we assume that any point we need can be labeled; if not, it is fixed, and we are done.

What we are intuitively doing with this labeling is picking a corner the point does not move closer to. We will eventually get a sequence of small 1-2-3 triangles converging to a point. If this point was not a fixed point, the continuity of f would suggest that as our triangles approach the point, all the corners would have the same labeling. This will be formalized as follows. First, we need to show that this labeling is in fact a Sperner labeling.

If we look at the corners, we note

- If $(1,0,0) \mapsto (a,b,c)$, $a < 1$ unless this is a fixed point, so this corner gets label 1.
- If $(0,1,0) \mapsto (a,b,c)$ then $a \geq 0$ but $b < 1$ (unless this is a fixed point), so this corner gets label 2.
- Similarly, $(0,0,1)$ gets label 3.

If we look at a point $(a,b,0)$ on the line connecting $(1,0,0)$ and $(0,1,0)$, we see that if $(a,b,0) \mapsto (c,d,e)$ then $c < a$ or $b < d$, so the label is 1 or 2. If not, we have $c \geq a, b \geq d$ so $c = a$, $d = b$ and $e = 0$ and the point is a fixed point.

Similarly, we can show that the points on the line connecting $(0,1,0)$ and $(0,0,1)$ get labels 2 or 3, and the points on the line connecting $(0,0,1)$ and $(1,0,0)$ get labels 3 or 1. Thus if we label a triangulation in this manner we get a Sperner labeling.

We now consider a sequence of triangulations with diameter going to 0. (The diameter of a triangulation is defined to be the maximum distance between adjacent vertices in a triangulation.) Each of these triangulations have at least one baby 1-2-3 triangle. Suppose these triangles have vertices

$$(x_{n,1}, y_{n,1}, z_{n,1})$$
$$(x_{n,2}, y_{n,2}, z_{n,2})$$
$$(x_{n,3}, y_{n,1}, z_{n,3})$$

with labels 1, 2 and 3 respectively. Here the n indicates that the triangle is from the n^{th} triangulation in the sequence of triangulations with diameter going to 0.

We now use the Bolzano-Weierstrass Theorem so find a convergent subsequence

$$(x_{n_k,i}, y_{n_k,i}, z_{n_k,i}) \to (x,y,z)$$

for $1 \leq i \leq 3$. We can do this for all three sequences at once because they are the corners of triangles whose diameters tend to 0. If we wanted to avoid the Bolzano-Weierstrass Theorem, an alternate approach is to use sub-triangulations and find nested 1-2-3 triangles.

Now if

$$(x_{n_k,i}, y_{n_k,i}, z_{n_k,i}) \mapsto (x'_{n_k,i}, y'_{n_k,i}, z'_{n_k,i})$$

and

$$(x,y,z) \mapsto (x', y', z')$$

we have (by the Sperner labeling)

$$x'_{n_k,1} \leq x_{n_k,1}$$

so by continuity

$$x' \leq x$$

Similarly, $y' \leq y$ and $z' \leq z$ so (x, y, z) is a fixed point. □

5. Closing Remarks

We have now proved a topological result using a combinatorial lemma about triangles! However, the student of algebraic topology may not be so surprised, because the topological proof of Brouwer's Fixed Point Theorem, and indeed a large part of algebraic topology, relies on a triangulating spaces. This result seems to be fundamentally connected with triangles.

Brouwer's Fixed Point Theorem is used in game theory to prove the existence of certain equilibrium, the theory of differential equations and other diverse areas.

References

[1] Gidea, Maria. The Brouwer Fixed Point Theorem. www.neiu.edu/~mathclub/brouwer.ppt

[2] Sperner's Lemma. PlanetMath. http://planetmath.org/encyclopedia/SpernersLemma.html

[3] Henle, Michael. A Combinatorial Introduction to Topology. W.H. Freeman and Company: San Francisco, 1979.

[4] Su, Francis Edward. Rental Harmony: Sperner's Lemma in Fair Division. American Mathematical Monthly, 106 (1999), 930-942. Available at http://www.math.hmc.edu/~su/papers.dir/rent.pdf