NOTES ON THE MIAN-CHOWLA SEQUENCE

FRANK CHU

ABSTRACT. The Mian-Chowla sequence $\{a_n\}_1^\infty$ is a B_2 sequence constructed using the *greedy* algorithm. Mian and Chowla [3] first conjectured that $a_n \leq n^2$. However, Stöhr proved the contrary: that $a_n \geq n^2$ for suitably large values of n. He also proved that $a_n \leq n^3$, and asked if $a_n = O(n^3)$ can be improved. In this paper, computational evidence will be provided to demonstrate that a_n does grow slower than n^3, and that $a_n \doteq O(n^{2.8})$ is more likely.

A B_2 sequence $A = \{a_n\}_1^\infty$ is an increasing sequence of positive integers such that for all $i \leq j$, the sums $a_i + a_j$ are all different. B_2 sequences were first used by Sidon in Fourier Analysis, and subsequently became what we now call Sidon sets. For any B_2 sequence A, and any integer n, define $\Omega(A, n)$ to be the unique integer k such that $a_k \leq n < a_{k+1}$, and $\Phi(n)$ to be the maximum of $\Omega(A, n)$. Erdős and Turán [2] showed that $\Phi(n) = \Theta(\sqrt{n})$. In fact, they proved that

$$(1) \qquad (\frac{1}{\sqrt{2}} - \epsilon)\sqrt{n} < \Phi(n) < (1 + \epsilon)\sqrt{n},$$

for any $\epsilon > 0$ and $n > n_0(\epsilon)$.

This means that a finite Sidon set $S \subset \{1, 2, \ldots, n\}$ has magniutde $|S| = \Theta(\sqrt{n})$. Finite Sidon sets have broad applications in radio-astronomy, X-ray crystallography, and the design of communication systems [1]. Since it is ususally desirable to use *dense* Sidon sets, many researchers attempt to construct Sidon sets that are as dense as possible. Singer [4] gave a construction that yields a Sidon set with $|S| > n^{\frac{1}{2}} - n^{\frac{5}{16}}$, so effectively we know $|S| \sim \sqrt{n}$. However, finding the *optimal* Sidon set, one that is the densest in a given range, is much more difficult. Moreover, most constructions cannot be extended to produce an infinite Sidon set (or a B_2 sequence). Thus, other than the information in (1), the growth rate of B_2 sequences remains quite mysterious.

There is a direct link between the growth rate of general B_2 sequences and equation (1). The upper bound in (1) is an improvement over the trivial bound $\Phi(n) < \sqrt{2n}$. Similarly, as the following lemma shows, the trivial lower bound $a_n > n(n-1)/2$ for the growth rate of B_2 sequences can be improved as well.

Lemma 1. *Suppose* $A = \{a_k\}_1^\infty$ *is a* B_2 *sequence. Then* $a_k > (1-\epsilon)k^2$ *for any* $\epsilon > 0$ *and* $k > k_0(\epsilon)$.

Proof. From the definition of Ω, and by (1), we get

$$k = \Omega(A, a_k) \leq \Phi(a_k) < (1+\epsilon)\sqrt{a_k}.$$

Hence we have

$$a_k > \frac{k^2}{1+\epsilon'}.$$

$\square$

On the other hand, the upper bound for the growth rate of any B_2 sequence can be translated from its lower bound for $\Omega(A, n)$. In particular, the lower bound in (1) is due to a construction yields a B_2 sequence A with $\Omega(A, n) > \sqrt{n/2}$, and so will satisfy $a_k < (2+\epsilon)k^2$. The proof is similar to that of Lemma 1.

A popular measure of compactness of infinite sequences is the *reciprocal sum.* Finding a dense Sidon set is similar to finding a B_2 sequence with a large reciprocal sum. Mathematicians define DDC (distinct difference constant) as the maximum reciprocal sum among all B_2 sequences. Computing the exact value of the DDC, and exhibiting a sequence achieving the DDC, is a well researched problem. A simple candidate for a B_2 sequence with a large reciprocal sum is the Mian-Chowla sequence, defined as follows:

Definition 2. *The Mian-Chowla sequence* [3] *is defined recursively as* $a_1 = 1$, *and choose* a_{m+1} *to be the smallest positive integer not in the set* $SD(m) = \{a_r + a_s - a_t \,|\, 1 \leq r, s, t \leq m\}$.

We first prove that the Mian-Chowla sequence is indeed a B_2 sequence. Then we give an alternative definition that is more convenient and widely used in the literature.

Lemma 3. *Suppose* $\{a_n\}_1^\infty$ *is the Mian-Chowla sequence. Then*

(i) *the sequence is a* B_2 *sequence, and*

(ii) *for all* $i < j$, *the differences* $a_j - a_i$ *are all different.*

Proof. We prove this by induction. The base case $A_1 = \{1\}$ is trivial. Suppose $A_m = \{a_1, ..., a_m\}$ satisfies (i) and (ii) above. Then by defintion of the sequence, $a_{m+1} \neq a_r + a_s - a_t$ for all $1 \leq r, s, t \leq m$. This means that (i) $a_{m+1} + a_t \neq a_r + a_s$, and (ii) $a_{m+1} - a_r \neq a_s - a_t$, for all $1 \leq r, s, t \leq m$. Hence $A_{m+1} = A_m \cup \{a_{m+1}\}$ satisfies properties (i) and (ii). □

Thus, the following more convenient definition of the Mian-Chowla sequence, which is a construction by the *greedy* algorithm, is equivalent to Definition 2.

Corollary 4. *The Mian-Chowla sequence can be constructed recursively by setting* $a_1 = 1$*, and choosing* a_{m+1} *to be the smallest positive integer such that the sums* $a_i + a_j$ *for all* $1 \leq i, j \leq m + 1$ *are all distinct. Alternatively, we can choose* a_{m+1} *so that the differences* $a_j - a_i$ *for* $1 \leq i < j \leq m + 1$ *are all different.*

It is long believed that the Mian-Chowla sequence reached the DDC. However, Zhang [7] disproved this by constructing a sequence that has a higher reciprocal sum. Interestingly, the Zhang sequence is just a slight modificaiton of the Mian-Chowla sequence, which only changes the 15th term of the Mian-Chowla sequence from 204 to 229, then continues with the greedy algorithm. Other sequences yielding large reciprocal sum also appear to be similar to the Mian-Chowla sequences [6]. Thus, it remains interesting to explore the growth of the Mian-Chowla sequence as an indicator to the growth of the optimal B_2 sequence.

Based on the definition, we can give some bounds for the Mian-Chowla sequence. First, being a B_2 sequence, the Mian-Chowla sequence will certainly satisfy the lower bound in Lemma 1. Stöhr [5] gave the following upper bound:

Theorem 5. *Suppose* $A = \{a_n\}$ *is the Mian-Chowla sequence. Then* $a_n \leq n^3$*. Hence* $a_n = O(n^3)$*.*

Proof. By definition, a_{m+1} is the smallest number not in the set $\{a_r + a_s - a_t \,|\, 1 \leq r, s, t \leq m\}$. The cardinality of this set is at most m^3, so it cannot cover all integers in the range $[1, m^3 + 1]$. In other words, there exists a number $\delta \leq m^3 + 1$ that is not in the set. Here, δ is a legitimate choice for a_{m+1}, and hence $a_{m+1} \leq m^3 + 1 < (m+1)^3$, which is what we want to show. □

In his paper, Stöhr asked if the bound $a_n = O(n^3)$ can be improved. To explore this question, we computed the Mian Chowla sequence up to the 6800th term, and plotted several graphs to analyze the growth of the sequence. The largest term we computed is about 3.2×10^9. The computer program we wrote to do the computation ran for approximately 112 hours on an AMD 2500+ machine. Figure 1 shows a plot of the Mian-Chowla sequence.

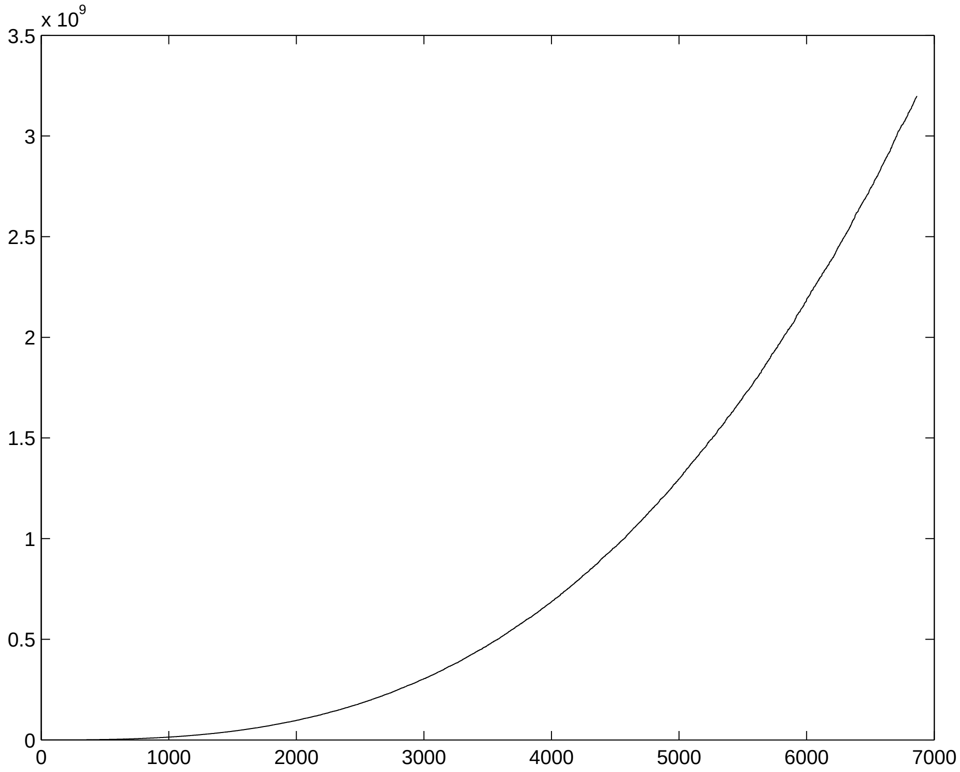

FIGURE 1. Plot of a_n for $1 \leq n \leq 6864$

Figure 2 shows the comparison between a_n and n^2. The left side shows a graph with small n values, with $0 \leq n \leq 30$. We can observe that when $n > 18$, the sequence starts to overtake n^2. The right shows a graph of with n reaching 6800. As we can see, a_n/n^2 grows steadily as n grows. We can infer from this that a_n grows much faster than n^2.

The benefits of computing more of the Mian-Chowla sequence is shown in Figure 3. A plot of comparison between a_n and polynomials of different powers shows clearly that a_n is rather unstable when $n < 500$, but follows a more fixed pattern when $n \geq 500$. This indicates to us that a good approximation of the growth of the Mian Chowla sequence probably has a dominant term that is a monomial. The power of this monomial can also be estimated from the graph.

When compared to $n^{2.7}$, a_n seems to grow more slowly. On the other hand, $a_n/n^{3.0}$ seems to decrease slightly as n increases. In fact, $a_n/n^{2.8}$ seems to be closest to being a constant function. So, we propose that $a_n = O(n^{2.8})$, or at least the dominant term in the estimation should be a monomial close to $n^{2.8}$.

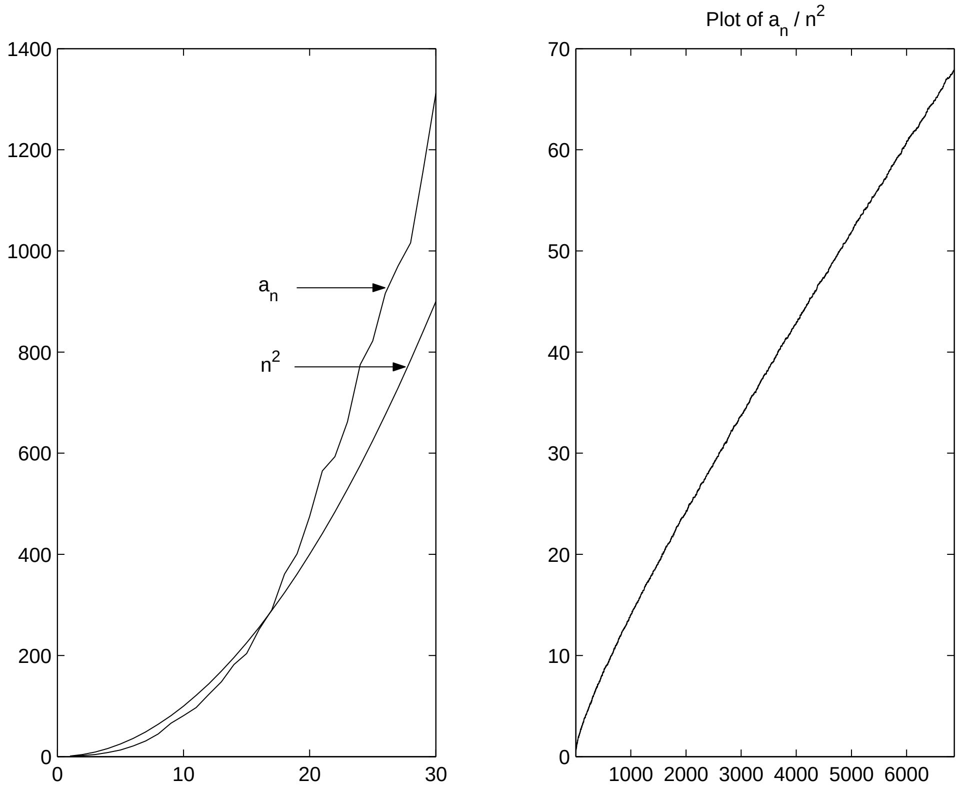

FIGURE 2. Comparison between a_n and n^2

To analyze the growth a little more, we give a log-log graph in Figure 4. If we suppose the estimation

$$a_n \doteq n^A$$

is a good one, then the plot

$$\frac{\log a_n}{\log n} \doteq A$$

will give us the estimation of the power of the monomial. Unfortunately, the findings are not as nice as what we wanted them to be. Although the graph shows clearly that the power of the monomial tends to a value greater than 2.5, it does not yield enough information for us to propose an estimated value. This inconsistency with Figure 3 is most likely due to a non-dominant factor in the growth estimation. This means that a_n grows more like $n^{2.8}f(n)$, where $f(n)$ is perhaps $\log n$, or other similar logarithm growth functions. At this time we do not have enough data to give a more precise estimation.

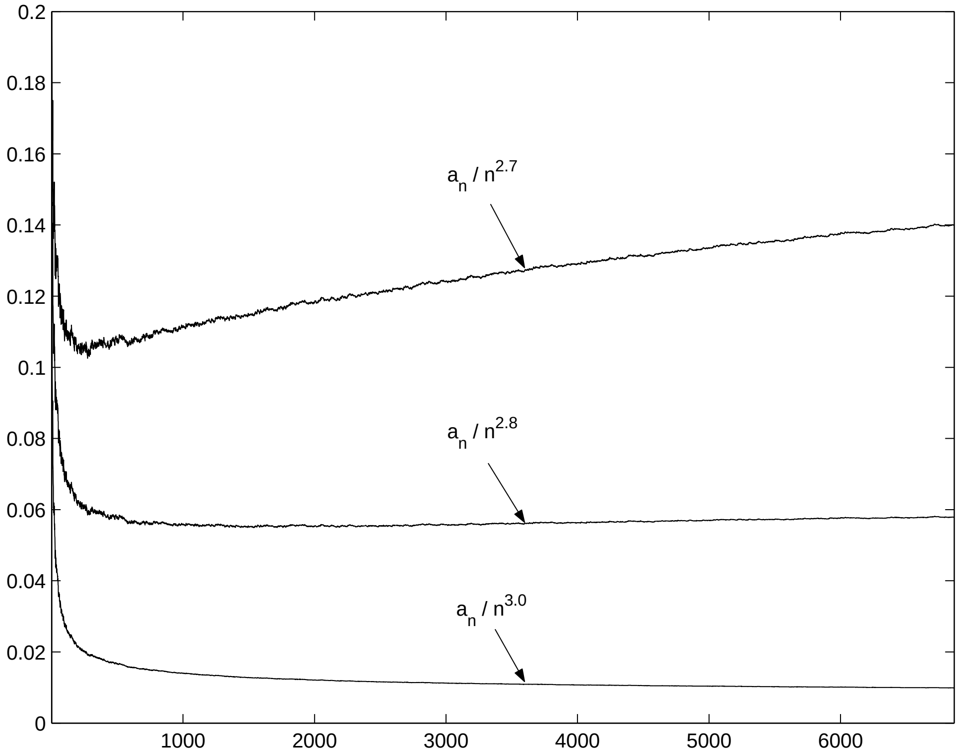

FIGURE 3. Several comparison plots

A distinct difference between the density measure of a set and the reciprocal sum of a sequence is that the reciprocal sum gives more emphasis on the lower vlaued terms in the sequence, while density is more uniform measurement. The Mian-Chowla sequence is certainlly not dense, as its growth is much greater than n^2. On the contrary, the accumulation of small values gives it a high reciprocal sum. According to our data, the growth rate of the sequence gets faster and faster as n increases, and stablizes when n is admissibily large. It would be interseting to see if one can construct a sequence which starts like the Mian-Chowla sequence, but continues half-way with a different algorithm that yields a denser sequence. Such sequences can possibly improve the DDC bound, as the best bounds currently are based on computing the reciprocal sum of the head of the sequence, and bounding the tail of the sequence analytically. When a_n is large, a dense set will match more closely to a sequence with high reciprocal sum.

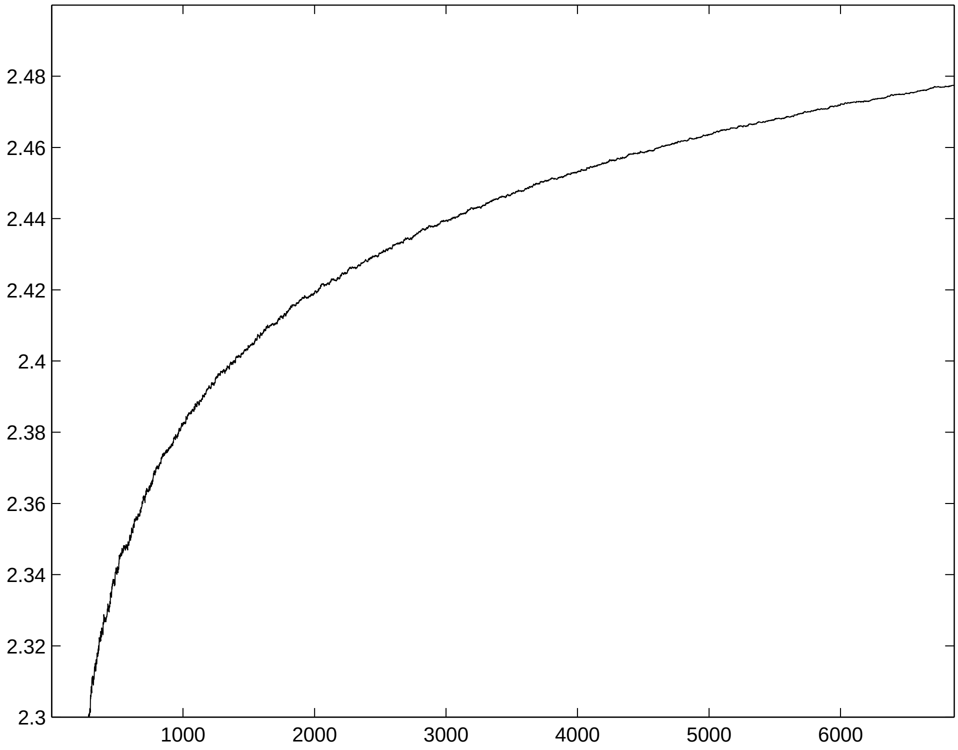

Figure 4. Plot of $\frac{\log a_n}{\log n}$

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