The Pythagorean Proposition

Marcus Emmanuel Barnes

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Abstract

The Pythagorean theorem is taught to schoolchildren around the world. One could say that the Pythagorean theorem is one of the first significant theorems that students learn. Not only is the theorem beautiful in its own right, it also leads the student to many other areas of mathematics, including trigonometry and analytic geometry. In celebration of this classical and useful result, three proofs of the Pythagorean theorem of different flavors will be presented.

Introduction

The genesis of this paper started one day while I was browsing through the geometry books in the Stacie Science Library at York University in Toronto. I came across a most curious book by a gentleman named Elisha Scott Loomis.

Loomis went to the effort to compile in one place proofs of the Pythagorean theorem. He was able to pack 370 different "demonstrations" of the Pythagorean theorem into one book, which happens to be called the "The Pythagorean Proposition". I dedicate this paper, based on my talk at the 2004 Canadian Undergraduate Mathematics Conference in Halifax, hosted by Dalhousie University, to the book,"The Pythagorean Proposition".

You may first remember the Pythagorean theorem being taught to you in elementary school around the concept of a variable was being introduced. The Pythagorean theorem is one of the first very powerful and elegant theorems that one encounters in elementary school mathematics (even if the theorem was not proven to you).

One could not do trigonometry without the Pythagorean Theorem, since one way of defining the trigonometric functions is based on the sides of a right angle triangle. The fundamental connections between the trigonometric functions are also derived using the Pythagorean theorem (for example, think of the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$). If you see a purported proof of the Pythagorean theorem using trigonometric methods, one should, to say the least, be skeptical of the validity of the proof.

Trigonometry leads one right into higher mathematics and physics, especially in areas where periodic phenomena must be described. The Pythagorean theorem is also a road into the study of analytic geometry, as one measures distances between points in the Euclidean plane with the Euclidean metric, that is, if you are given two points (x_1, x_2) and (y_1, y_2), the distance between these two points is $\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$.

It would be hard to argue otherwise that the Pythagorean theorem is among the most celebrated mathematical propositions of all time, having come down to us from antiquity. One can picture a scenario where two young school children are on a school bus arguing over who is smarter:

Child 1: "Oh yeah! $E = mc^2$."

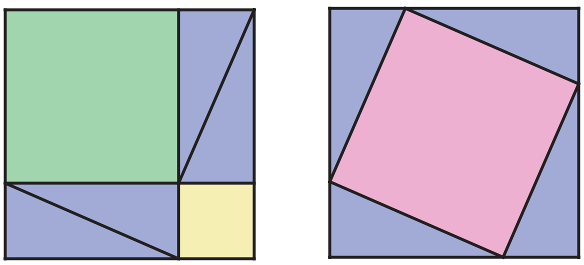
Child 2: "Well, $a^2 + b^2 = c^2$. So there!"

It would be a safe bet to assume that Child 1 does not know the meaning of Einstein's famous equation relating the mass of an object with the amount of energy it contains. A better bet would be to assume that Child 2 knows the meaning of $a^2 + b^2 = c^2$, an equation that has more practical applications to everyday life. Child 2 going forth into the world with the Pythagorean theorem in her tool belt of mathematical skills would most likely live a more comfortable life than her school mate, Child 2.

Now we will get to the heart of this paper. In celebration of the Pythagorean theorem, three proofs of different natures will be presented. The first proof will be a "visual proof"; the second will be an algebraic proof; the third, a geometric proof. Note that the classification of the "algebraic" and "geometric" proofs respectively, is the classification given to them by Loomis in his book [1].

The Visual 'Proof'

The word proof in the the title of this section is in scare-quotes for those readers who have qualms about visual proofs. The proof given here is adapted from the *Chou pei suan ching* as presented in [2]. The *Chou pei suan ching* is one of the earliest textual sources for the history of Chinese mathematics that is known to exist. The author of the proof is unknown, but the author probably lives some time during the years from around 500 to 200 B.C.E. (See Figure 1).



a b

Figure 1

A short glance at the visual proof may have already shown you the validity of the Pythagorean theorem. If you have trouble deciphering the proof, observe the following. Square a is supposed to be dissected into 4 congruent right-angled triangles (the blue regions) plus the squares on the sides of the legs on one of the triangles (the green and yellow inner regions). Now square b is supposed to be congruent to square a. Square b is dissected into 4 congruent right-angled triangles (the blue regions) that are congruent to the 4 congruent right-angled triangles in square a, plus the square on the hypotenuse of one of the 4 congruent right-angled triangles. The areas of square a and square b are equal because square a and square b are congruent. Subtracting the congruent triangles from both square a and square b leaves us with the squares on the legs of one of the 4 congruent triangles (the yellow and green squares) in what used to be square a and with the square on the hypotenuse (the magenta square) in what used to be square b. The regions left in what used to be square a must be equal to the region left in what used to be square b. Thus we have proven that for a right-angled triangle, the squares on the legs of the triangle are equal to the square on the hypotenuse. This is the Pythagorean theorem.

The Algebraic Proof

The algebraic proof presented here is proof number 38 from Loomis' book [1]. Refer to Figure 2 in what follows.

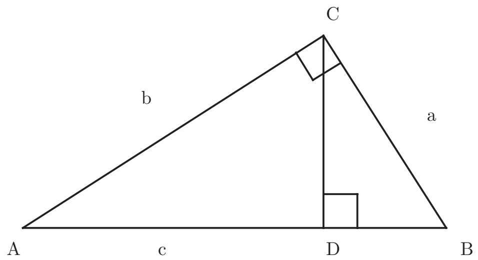


Figure 2

We have by similar triangles that

$$\frac{AD}{AC} = \frac{AC}{AB}.$$

So, $AC^2 = AD \cdot AB$. We also have by similar triangles that

$$\frac{BD}{BC} = \frac{BC}{BA}.$$

Hence, $BC^2 = BD \cdot BA$. Therefore,

$$AC^2 + BC^2 = AD \cdot AB + BD \cdot BA = (AD + DB) \cdot AB = AB^2.$$

In conclusion, we have $a^2 + b^2 = c^2$ and our proof is complete.

The Geometric Proof

We now come to the last proof of this paper. The geometric proof presented presently is proof number 220 from Loomis' book [1]. Refer to Figure 3 in what follows.

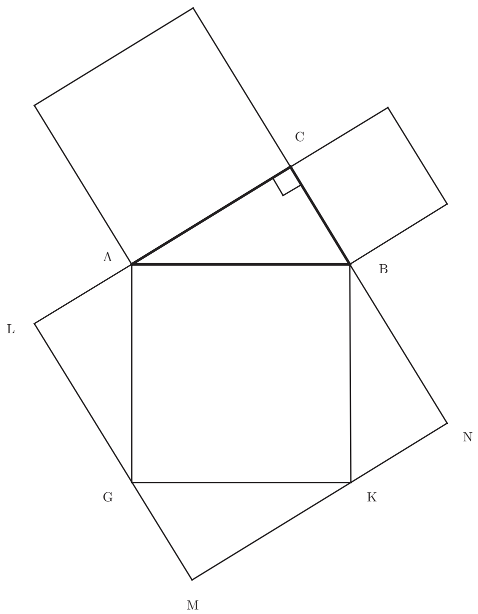


Figure 3

Given a right-angled triangle $\triangle ABC$ and the square on its hypotenuse, draw a line parallel to CB through G and a line parallel to AC through K. Mark as M the point where these two lines meet. Now extend AC to meet the line MG at L and extend CB to meet the line MK at N. We now have a square $CLMN$. Next observe that

$$\begin{aligned} ABKG &= CLMN - 4 \cdot \frac{1}{2} \cdot CB \cdot CA \\ &= (CB + CA)^2 - 2 \cdot CB \cdot CA \\ &= CB^2 + 2 \cdot CB \cdot CA + CA^2 - 2 \cdot CB \cdot CA. \end{aligned}$$

Note that $ABKG = AB^2$. Therefore $AB^2 = BC^2 + CA^2$, thus proving the Pythagorean theorem.

Conclusion

Admittedly, the content of this paper was rather elementary for a paper that was presented at an undergraduate conference. However, as this paper was presented during the first conference slot on the morning of the first day of the conference, it served the purpose of waking up the conference participants gently and getting them ready for the rest of an enjoyable and successful conference. Besides, who can resist a nice theorem with many elementary and elegant proofs — not me.

References

[1] LOOMIS, ELISH SCOTT, The Pythagorean Proposition: Its Demonstration Analyzed and Classified and Bibliography of Sources for Data of the Four Kinds of 'Proofs', National Council of Teachers of Mathematics, Washington, DC, 1968.

[2] NELSON, ROGER B., Proof Without Words: Exercises in Visual Thinking, The Mathematical Association of Amercica (Incorpertated), Washington, DC, 1993.

Contact Information:

Marcus Emmanuel Barnes
c/o Prof. Richard Ganong,
Department of Mathematics and Statistics,
York University,
N520 Ross Building,
4700 Keele Street,
Toronto, Ontario, Canada M3J 1P3
E-mail: yu250248@mathstat.yorku.ca